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Four-dimensional $N = 4$, $SO(4)$ Gauged Supergravity from $D = 11$

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ABSTRACT

We construct the complete and explicit non-linear Kaluza-Klein Ansatz for deriving the bosonic sector of the standard $N = 4$ $SO(4)$ gauged four-dimensional supergravity from the reduction of $D = 11$ supergravity on S^7. This provides a way of interpreting all bosonic solutions of the four-dimensional gauged theory as exact solutions in eleven-dimensional supergravity. We discuss certain limiting forms of the Kaluza-Klein reduction, and compare them with related forms in the Freedman-Schwarz $N = 4$ $SU(2) \times SU(2)$ gauged theory. This leads us to the result that the Freedman-Schwarz model is in fact a singular limiting case of the standard $SO(4)$ gauged supergravity. We show that in this limit, our Ansatz for getting the $SO(4)$ gauged theory as an S^7 reduction from $D = 11$ indeed reduces to an $S^3 \times S^3$ reduction from $D = 10$, which makes contact with previous results in the literature. We also show that there is no distinction to be made between having equal or unequal values for the gauge coupling constants g and $\tilde{g}$ of the two $SU(2)$ gauge-group factors in the standard $N = 4$ $SO(4)$ gauged supergravity, whilst by contrast the ratio of g to $\tilde{g}$ is a non-trivial parameter of the Freedman-Schwarz model.

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1 Introduction

In the conjectured AdS/CFT correspondence [1, 2, 3], it becomes important to establish how the lower-dimensional gauged supergravities arise through spherical Kaluza-Klein reductions of the fundamental theories in $D = 10$ or $D = 11$. It has long been known in certain cases that at the level of linearised fluctuations around an AdS $\times$ Sphere background, the massless excitations in the Kaluza-Klein spectrum describe the maximal gauged supergravity multiplet. The cases where this occurs include $D = 11$ supergravity compactified on S^7 [4] or S^4 [5], and type IIB supergravity compactified on S^5 [6, 7].

What is much less clear is whether these results extend nicely beyond the level of the linearised analysis. It is obvious that if one performs expansions of all the fields in terms of complete sets of harmonics on the sphere, then one will necessarily obtain a lower-dimensional theory comprising the gauged supergravity coupled to an infinite tower of massive multiplets. *A priori*, one might expect that beyond the linearised level, there could be couplings of the form $H\,L^2$, $H\,L^3$, *etc.* in the lower-dimensional Lagrangian, where H represents a heavy field and L a massless one. Such couplings would prevent one from rigorously setting the heavy fields to zero, since the massless fields would be acting as sources for them. Such a phenomenon does not happen in a toroidal reduction, since the torus harmonics associated with the massless modes are constants, while those associated with the massive modes depend on the torus coordinates. Thus it is guaranteed in that case that no non-linear products of zero-mode harmonics can generate non-zero-mode harmonics. The truncation to the massless sector is therefore guaranteed to be consistent in a toroidal reduction.

On the sphere, the harmonics associated with the massless fields can depend on the coordinates of the sphere (for example, the Killing vectors associated with the massless gauge bosons), and so it is far from obvious that once the non-linear interactions are included, there will be no couplings linear in heavy fields, of the kind we discussed above. Quite the contrary, in fact; it is easy to see that in general such terms *will* be present, and so a generic theory reduced on a sphere cannot be consistently truncated to the massless sector. Remarkably, however, it turns out that these consistency problems are avoided in the case of the sphere reductions of $D = 11$ supergravity. For the S^7 reduction, indications of this were seen in [8, 9], and a complete demonstration of the consistency of the truncation was given in [10]. For the S^4 reduction, the explicit reduction Ansatz was recently constructed [11], again showing that the truncation to the massless sector is consistent. No analogous result has been derived for the S^5 reduction of type IIB supergravity, but it is strongly believed to work there too.

The Kaluza-Klein Ansätze for the complete S^7 and S^4 reductions are rather complicated (the S^7 case is especially complicated, and indeed the reduction scheme obtained in [10] is somewhat implicit, which is presumably inevitable since the $N = 8$ gauged theory is itself intrinsically rather complicated). In a number of recent papers, completely explicit consistent reduction Ansätze have been constructed for various further (consistent) truncations of the maximal gauged supergravities. The advantage of looking at these smaller theories is that the expressions for the Ansätze become much more manageable, and it becomes possible to present fully explicit results. These results are completely sufficient if one is interested in knowing how to embed lower-dimensional solutions that use only the truncated subset of fields into the original theory in $D = 10$ or $D = 11$. Cases that have been worked out in this way include truncations to the maximal abelian subgroups $U(1)^4$, $U(1)^3$ and $U(1)^2$ of the full $SO(8)$, $SO(6)$ and $SO(5)$ gauge groups in $D = 4$, 5 and 7 [12]. Cases with surviving non-abelian gauge groups have also been constructed; the $N = 1$ $SU(2)$ gauged supergravity in $D = 7$ [13], and the $N = 4$ gauged $SU(2) \times U(1)$ supergravity in $D = 5$ [14]. The former arises from an S^4 reduction from $D = 11$, while the latter comes from an S^5 reduction from type IIB supergravity. Another case that has been obtained is $N = 2$ $SU(2)$ gauged supergravity in $D = 6$ [15]. This is in fact the largest gauged theory that exists in $D = 6$ [16], even though ungauged supergravity with $N = 4$ exists in $D = 6$. The six-dimensional theory arises from a local S^4 reduction of massive type IIA supergravity [15].[1] An explicit Ansatz for the embedding of the symmetric scalar potential of $D = 5$ gauged supergravity into the metric of type IIB was also obtained, in [17] (see also [18]). This was extended to the full consistent embedding, giving the Ansatz also for the 5-form antisymmetric tensor (the only other active field in this truncation) in [19]. The results were also extended to the full consistent reduction Ansätze for the symmetric scalar potentials of $D = 7$ and $D = 4$ gauged supergravities from $D = 11$, and for the analogous scalar potential in $D = 6$ gauged supergravity from massive type IIA, in [19].

The consistency of the reduction is of particular importance in the context of the AdS/CFT correspondence. One is interested in considering p-brane configurations in the higher dimension that carry a large charge N, in the limit when $N \longrightarrow \infty$. From the lower-dimensional point of view, this corresponds to configurations such as charged AdS black holes where the gauge fields that support the solution take large values. If, heuristically speaking, the massless fields denoted by L are taking very large values then it is crucial

[1] In all cases, attention has been focussed on the bosonic sectors of the supergravities, since these are the fields that participate in p-brane solutions.

that there should be no $H\,L^2$, $H\,L^3$, *etc.* , couplings in the theory, in order that the neglect of the massive fields H can be justified.

In this paper, we shall construct another example where an explicit consistent reduction can be obtained. We consider the case of $N = 4$ $SO(4)$ gauged supergravity in $D = 4$.[2] Of course in principle the reduction Ansatz for this theory should be subsumed in the $N = 8$ reduction described in [10]. In practice, as we have indicated, the results in [10] are somewhat implicit, and furthermore the full results for the reduction of the $D = 11$ 4-form field strength are not presented there. (The metric reduction Ansatz, on the other hand, *is* given explicitly, and in fact we make use of the metric reduction given in [10] in obtaining our results.) This $N = 4$ example is rather more complicated than previous ones that have been explicitly considered. In particular, the bosonic field content includes not only a dilaton but also an axion, and this leads to a more involved structure in the reduction Ansätze. As usual, the bulk of the complexity in determining the reduction Ansätze centres around the antisymmetric tensor fields.

In section 2 we present our results for the consistent Kaluza-Klein reduction Ansatz, including a discussion of the geometry of the internal 7-sphere. In section 3 we present the Lagrangian and the equations of motion for the four-dimensional $N = 4$ $SO(4)$ gauged supergravity, and in section 4 we discuss how our Kaluza-Klein Ansatz produces this theory as an exact embedding in $D = 11$ supergravity. In section 5, we discuss certain singular limits of the reduction, and we compare them with related limits in the Freedman-Schwarz $N = 4$ $SU2) \times SU(2)$ gauged supergravity. We show that in fact the Freedman-Schwarz model can be understood as a singular limit of the standard $N = 4$ $SO(4)$ gauged theory, in which the axion is shifted by an infinite constant. The S^7 internal space degenerates to $\mathbb{R} \times S^3 \times S^3$ in this limit. We also show that in the standard $N = 4$ $SO(4)$ gauged theory, there is no distinction to be made between the cases of equal or unequal gauge coupling constants g and $\tilde{g}$ for the two $SU(2)$ factors in the gauge group, since one can make rescalings that allow the ratio to be adjusted at will. (This observation was also made in [24, 23].) By contrast, no such rescalings are possible in the Freedman-Schwarz model, and so there the ratio $g/\tilde{g}$ is a non-trivial parameter of the theory. After concluding remarks, we present details in Appendix A of the derivation of the metric reduction Ansatz.

[2] We should emphasise that here we are, intitially, discussing the *standard* $N = 4$ $SO(4)$ gauged theory of [20], not the Freedman-Schwarz $N = 4$ $SU(2) \times SU(2)$ gauged theory of [21]. In section 5, however, we show that the latter is a singular limit of the former.

2 The Ansatz

In this section, we present our results for the Kaluza-Klein reduction Ansatz for obtaining $N=4$ $SO(4)$ gauged supergravity in $D=4$ from an S^7 reduction of $D=11$ supergravity. Some of the details of how we arrived at this Ansatz are discussed in Appendix A. For the metric, we find

$$d\hat{s}_{11}^2 = \Delta^{\frac{2}{3}}\, ds_4^2 + 2g^{-2}\,\Delta^{\frac{2}{3}}\, d\xi^2 + \tfrac{1}{2}g^{-2}\,\Delta^{\frac{2}{3}} \Big[\frac{c^2}{c^2\, X^2 + s^2}\,\sum_i (h^i)^2 + \frac{s^2}{s^2\,\widetilde{X}^2 + c^2}\,\sum_i (\tilde{h}^i)^2\Big]\,, \quad (1)$$

where

$$\begin{aligned} &\widetilde{X} \equiv X^{-1}\, q\,, \qquad q^2 \equiv 1 + \chi^2\, X^4\,,\\ &\Delta \equiv \Big[(c^2\, X^2 + s^2)(s^2\,\widetilde{X}^2 + c^2)\Big]^{\frac{1}{2}}\,,\\ &c \equiv \cos\xi\,, \qquad s \equiv \sin\xi\,,\\ &h^i \equiv \sigma_i - g\, A_{(1)}^i\,, \qquad \tilde{h}^i \equiv \tilde{\sigma}_i - g\,\widetilde{A}_{(1)}^i\,. \end{aligned} \quad (2)$$

The three quantities σ_i are left-invariant 1-forms on $S^3 = SU(2)$, and the three $\tilde{\sigma}_i$ are left-invariant 1-forms on a second S^3. They satisfy

$$d\sigma_i = -\tfrac{1}{2}\epsilon_{ijk}\,\sigma_j\wedge\sigma_k\,, \qquad d\tilde{\sigma}_i = -\tfrac{1}{2}\epsilon_{ijk}\,\tilde{\sigma}_j\wedge\tilde{\sigma}_k\,. \quad (3)$$

The $SU(2)$ Yang-Mills potentials $A_{(1)}^i$, together with the second set $\widetilde{A}_{(1)}^i$, together comprise the $SO(4)\sim SU(2)\times SU(2)$ gauge group of the $N=4$ gauged supergravity in $D=4$. The constant g is the gauge coupling constant. The remaining bosonic fields of the $N=4$ supermultiplet are the dilaton ϕ and the axion χ. The dilaton parameterises the quantity X appearing in (1) and (2), being related to it by

$$X = e^{\frac{1}{2}\phi}\,. \quad (4)$$

We find that the Ansatz for $\hat{F}_{(4)}$ is as follows:

$$\hat{F}_{(4)} = -g\,\sqrt{2}\,U\,\epsilon_{(4)} - \frac{4s\,c}{g\,\sqrt{2}}\,X^{-1}\,{*dX}\wedge d\xi + \frac{\sqrt{2}s\,c}{g}\,\chi\,X^4\,{*d\chi}\wedge d\xi + \hat{F}'_{(4)} + \hat{F}''_{(4)}\,, \quad (5)$$

where

$$U = X^2\,c^2 + \widetilde{X}^2\,s^2 + 2\,, \quad (6)$$

and $\hat{F}'_{(4)} = d\hat{A}'_{(3)}$, with

$$\hat{A}'_{(3)} = f\,\epsilon_{(3)} + \tilde{f}\,\tilde{\epsilon}_3\,, \quad (7)$$

where $\epsilon_{(3)} = \frac{1}{6}\epsilon_{ijk}\, h^i\wedge h^j\wedge h^k$ and $\tilde\epsilon_{(3)} = \frac{1}{6}\epsilon_{ijk}\, \tilde h^i\wedge \tilde h^j\wedge \tilde h^k$. The functions f and $\tilde f$ are given by

$$
\begin{aligned}
f &= \frac{1}{2\sqrt2}\, g^{-3}\, c^4 \chi\, X^2\, (c^2\, X^2 + s^2)^{-1}\,,\\
\tilde f &= -\frac{1}{2\sqrt2}\, g^{-3}\, s^4\, \chi\, X^2\, (s^2\, \widetilde X^2 + c^2)^{-1}\,.
\end{aligned}
\tag{8}
$$

The field strength contribution $\hat F'_{(4)}$ is therefore given by

$$
\begin{aligned}
\hat F'_{(4)} &= \frac{\partial f}{\partial\chi}\, d\chi\wedge\epsilon_{(3)} + \frac{\partial f}{\partial X}\, dX\wedge\epsilon_{(3)} + \frac{\partial f}{\partial\xi}\, d\xi\wedge\epsilon_{(3)}\\
&\quad + \frac{\partial\tilde f}{\partial\chi}\, d\chi\wedge\tilde\epsilon_{(3)} + \frac{\partial\tilde f}{\partial X}\, dX\wedge\tilde\epsilon_{(3)} + \frac{\partial\tilde f}{\partial\xi}\, d\xi\wedge\tilde\epsilon_{(3)}\\
&\quad -\tfrac12 f\, g\, \epsilon_{ijk}\, h^i\wedge h^j\wedge F^k_{(2)} - \tfrac12 \tilde f\, g\, \epsilon_{ijk}\, \tilde h^i\wedge\tilde h^j\wedge \widetilde F^k_{(2)}\,.
\end{aligned}
\tag{9}
$$

The terms in $\hat F''_{(4)}$ comprise those involving the $SU(2)\times SU(2)$ Yang-Mills field strengths $F^i_{(2)}$ and $\widetilde F^i_{(2)}$. These are given by

$$
\begin{aligned}
\sqrt2\, \hat F''_{(4)} &= g^{-2}\, s\, c\, X^{-2}\, d\xi\wedge h^i\wedge {*F^i_{(2)}} + \tfrac14 g^{-2}\, c^2\, X^{-2}\, \epsilon_{ijk}\, h^i\wedge h^j\wedge {*F^k_{(2)}}\\
&\quad -g^{-2}\, s\, c\, \widetilde X^{-2}\, d\xi\wedge \tilde h^i\wedge {*\widetilde F^i_{(2)}} + \tfrac14 g^{-2}\, s^2\, \widetilde X^{-2}\, \epsilon_{ijk}\, \tilde h^i\wedge\tilde h^j\wedge {*\widetilde F^k_{(2)}}\,,\\
&\quad +g^{-2}\, s\, c\, \chi\, d\xi\wedge h^i\wedge F^i_{(2)} + \tfrac14 g^{-2}\, c^2 \chi\, \epsilon_{ijk}\, h^i\wedge h^j\wedge F^k_{(2)}\,,\\
&\quad +g^{-2}\, s\, c\, \chi\, X^2\, \widetilde X^{-2}\, d\xi\wedge\tilde h^i\wedge \widetilde F^i_{(2)} - \tfrac14 g^{-2}\, s^2\, \chi\, X^2\, \widetilde X^{-2}\, \epsilon_{ijk}\, \tilde h^i\wedge\tilde h^j\wedge\widetilde F^k_{(2)}\,.
\end{aligned}
\tag{10}
$$

For the purposes of verifying the consistency of the Ansatz, it is useful to record that the eleven-dimensional Hodge dual of $\hat F_{(4)}$ is given by

$$
\begin{aligned}
\hat{*}\hat F_4 &= \tfrac14 g^{-6}\, s^3\, c^3\, \Delta^{-2}\, U\, d\xi\wedge\epsilon_{(3)}\wedge\tilde\epsilon_{(3)} - \tfrac14 g^{-6}\, s^4\, c^4\, \Delta^{-2}\, X^{-1}\, dX\wedge\epsilon_{(3)}\wedge\tilde\epsilon_{(3)}\\
&\quad +\tfrac18 g^{-6}\, s^4\, c^4\, \Delta^{-2}\, X^4\, \chi\, d\chi\wedge\epsilon_{(3)}\wedge\tilde\epsilon_{(3)} + \hat{*}\hat F'_{(4)} + \hat{*}\hat F''_{(4)}\,,
\end{aligned}
\tag{11}
$$

where the term $\hat{*}\hat F'_{(4)}$ is given by

$$
\begin{aligned}
\hat{*}\hat F'_{(4)} &= -\sqrt2\, g^{-1}\, s^3\, c^{-3}\, \frac{\partial f}{\partial\chi}\, \Delta^{-2}\, \Omega^3\, {*d\chi}\wedge d\xi\wedge\tilde\epsilon_{(3)}\\
&\quad -\sqrt2\, g^{-1}\, s^3\, c^{-3}\, \frac{\partial f}{\partial X}\, \Delta^{-2}\, \Omega^3\, {*dX}\wedge d\xi\wedge\tilde\epsilon_{(3)}\\
&\quad +\sqrt2\, g^{-1}\, c^3\, s^{-3}\, \frac{\partial \tilde f}{\partial\chi}\, \Delta^{-2}\, \widetilde\Omega^3\, {*d\chi}\wedge d\xi\wedge\epsilon_{(3)}\\
&\quad +\sqrt2\, g^{-1}\, c^3\, s^{-3}\, \frac{\partial \tilde f}{\partial X}\, \Delta^{-2}\, \widetilde\Omega^3\, {*dX}\wedge d\xi\wedge\epsilon_{(3)}\\
&\quad +\tfrac{1}{\sqrt2}\, g\, s^3\, c^{-3}\, \frac{\partial f}{\partial\xi}\, \Delta^{-2}\, \Omega^3\, \epsilon_{(4)}\wedge\tilde\epsilon_{(3)} - \tfrac{1}{\sqrt2}\, g\, c^3\, s^{-3}\, \frac{\partial\tilde f}{\partial\xi}\, \Delta^{-2}\, \widetilde\Omega^3\, \epsilon_{(4)}\wedge\epsilon_{(3)} \qquad (12)\\
&\quad -\tfrac{1}{\sqrt2}\, f\, g^{-2}\, s^3\, c^{-1}\, \Omega\, \widetilde\Omega^{-1}\, d\xi\wedge h^i\wedge {*F^i_{(2)}}\wedge\tilde\epsilon_{(3)}\\
&\quad +\tfrac{1}{\sqrt2}\, \tilde f\, g^{-2}\, c^3\, s^{-1}\, \widetilde\Omega\, \Omega^{-1}\, d\xi\wedge\tilde h^i\wedge {*\widetilde F^i_{(2)}}\wedge\epsilon_{(3)}\,,
\end{aligned}
$$

and we have defined

$$\Omega \equiv c^2\, X^2 + s^2\,, \qquad \widetilde\Omega \equiv s^2\, \widetilde X^2 + c^2\,. \tag{13}$$

The final term in (11) is given by

$$\begin{aligned}
\hat{*}\hat F''_{(4)} \;=\;& -\tfrac{1}{16} g^{-5}\, s^4\, c^2\, \widetilde\Omega^{-1}\, X^{-2}\, \epsilon_{ijk}\, h^j\wedge h^k \wedge F^k_{(2)}\wedge \tilde\epsilon_{(3)}\\
& -\tfrac14 g^{-5}\, s^3\, c\, \Omega\, \widetilde\Omega^{-1}\, X^{-2}\, d\xi\wedge h^i\wedge F^i_{(2)}\wedge\tilde\epsilon_3\\
& -\tfrac{1}{16} g^{-5}\, s^2\, c^4\, \Omega^{-1}\, \widetilde X^{-2}\, \epsilon_{ijk}\, \tilde h^j\wedge \tilde h^k\wedge \widetilde F^k_{(2)}\wedge\epsilon_{(3)}\\
& +\tfrac14 g^{-5}\, s\, c^3\, \widetilde\Omega\, \Omega^{-1}\, \widetilde X^{-2}\, d\xi\wedge \tilde h^i\wedge \widetilde F^i_{(2)}\wedge\epsilon_3\\
& +\tfrac{1}{16} g^{-5}\, s^4\, c^2\, \widetilde\Omega^{-1}\, \chi\, \epsilon_{ijk}\, h^j\wedge h^k\wedge {*F}^k_{(2)}\wedge\tilde\epsilon_{(3)}\\
& +\tfrac14 g^{-5}\, s^3\, c\, \Omega\, \widetilde\Omega^{-1}\, \chi\, d\xi\wedge h^i\wedge {*F}^i_{(2)}\wedge\tilde\epsilon_3\\
& -\tfrac{1}{16} g^{-5}\, s^2\, c^4\, \Omega^{-1}\, \chi\, X^2 \widetilde X^{-2}\, \epsilon_{ijk}\, \tilde h^j\wedge\tilde h^k\wedge {*\widetilde F}^k_{(2)}\wedge\epsilon_{(3)}\\
& +\tfrac14 g^{-5}\, s\, c^3\, \widetilde\Omega\, \Omega^{-1}\, \chi\, X^2\, \widetilde X^{-2}\, d\xi\wedge\tilde h^i\wedge {*\widetilde F}^i_{(2)}\wedge\epsilon_3
\end{aligned} \tag{14}$$

A number of remarks about the reduction Ansatz are in order. First, we note that there is a residual Z_2 subgroup of the original global $SL(2,\mathbb{R})$ symmetry of the ungauged theory, under which the various quantities are mapped to their primed images, given by

$$\begin{aligned}
& X' = \widetilde X\,, \qquad \chi'\, X'^2 = -\chi\, X^2\,, \qquad {A'}^i_{(1)} = \widetilde A^i_{(1)}\,, \qquad {\tilde A'}^i_{(1)} = A^i_{(1)}\,,\\
& c' = s\,, \qquad s' = -c\,, \qquad h_i' = \tilde h_i\,, \qquad \tilde h_i' = h_i\,,\\
& \epsilon'_{(3)} = \tilde\epsilon_{(3)}\,, \qquad \tilde\epsilon'_{(3)} = \epsilon_{(3)}\,.
\end{aligned} \tag{15}$$

In particular, we have $\Delta' = \Delta$, $q' = q$ and $U' = U$. In fact, the entire Ansatz for the metric and 4-form is invariant under the Z_2. It corresponds to an interchange of the two 3-spheres in our description of S^7 as a foliation of $S^3\times S^3$. Correspondingly, in the four-dimensional theory itself, the Z_2 symmetry involves an interchange of the two $SU(2)$ gauge fields.

The geometry of the internal 7-sphere can be understood as follows. If we look at the metric Ansatz (1) in the "unexcited" state where the gauge fields, axion and dilaton vanish (and so $X = \widetilde X = 1$), we see that up to a constant factor of $\frac12 g^{-2}$ the internal 7-dimensional metric becomes

$$d\Omega_7^2 = d\xi^2 + \cos^2\xi\, d\Omega_3^2 + \sin^2\xi\, d\widetilde\Omega_3^2\,, \tag{16}$$

where $d\Omega_3^2 = \frac14\sum_i \sigma_i^2$ and $d\widetilde\Omega_3^2 = \frac14 \sum_i \tilde\sigma_i^2$ are metrics on the two unit 3-spheres. In fact $d\Omega_7^2$ is a metric on the unit 7-sphere, with the "latitude" coordinate ξ running between the limits $0\le\xi\le\frac12\pi$, at which one or other of the two 3-spheres shrinks to zero radius. This geometrical description of the 7-sphere is analogous to the description of a S^3 as a foliation of Clifford tori $S^1\times S^1$, in which one has $d\Omega_3^2 = d\xi^2 + \cos^2\xi\, d\phi_1^2 + \sin^2\xi\, d\phi_2^2$.

3 $N=4$ $SO(4)$ gauged four-dimensional supergravity

The bosonic Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_4 &= R\,{*\mathbb{1}} - \tfrac12 {*d\phi}\wedge d\phi - \tfrac12 e^{2\phi}\,{*d\chi}\wedge d\chi - V\,{*\mathbb{1}} \\
&\quad -\tfrac12 e^{-\phi}\,{*F^i_{(2)}}\wedge F^i_{(2)} - \tfrac12 \frac{e^{\phi}}{1+\chi^2\,e^{2\phi}}\,{*\widetilde F^i_{(2)}}\wedge \widetilde F^i_{(2)}\,, \\
&\quad -\tfrac12 \chi\,F^i_{(2)}\wedge F^i_{(2)} + \tfrac12 \frac{\chi\,e^{2\phi}}{1+\chi^2\,e^{2\phi}}\,\widetilde F^i_{(2)}\wedge \widetilde F^i_{(2)}\,,
\end{aligned}\tag{17}$$

where the potential V is

$$V = -2g^2\,(4+2\cosh\phi + \chi^2\,e^{\phi})\,,\tag{18}$$

and

$$F^i_{(2)} = dA^i_{(1)} + \tfrac12 g\,\epsilon_{ijk}\,A^j_{(1)}\wedge A^k_{(1)}\,,\qquad \widetilde F^i_{(2)} = d\widetilde A^i_{(1)} + \tfrac12 g\,\epsilon_{ijk}\,\widetilde A^j_{(1)}\wedge \widetilde A^k_{(1)}\,.\tag{19}$$

(We have chosen to set the two gauge couplings for the two $SU(2)$ gauge groups equal here. There is no loss of generality involved in doing this; one can always restore the two coupling constants by shifting ϕ by a constant, accompanied by appropriate rescalings of χ and the gauge potentials. We shall return to this point in section 5.)

The dilaton ϕ is related to the quantity X of the previous section by

$$X = e^{\frac12\phi}\,.\tag{20}$$

The equations for motion for X and χ that follow from (17) are:

$$\begin{aligned}
d(X^{-1}\,{*dX}) &= -\tfrac12 X^4\,{*d\chi}\wedge d\chi + g^2\,(X^2 - X^{-2} + \chi^2\,X^2)\,\epsilon_{(4)} + \tfrac14 X^{-2}\,{*F^i_{(2)}}\wedge F^i_{(2)} \\
&\quad -\tfrac14(1-\chi^2\,X^4)\,X^2\,q^{-4}\,{*\widetilde F^i_{(2)}}\wedge \widetilde F^i_{(2)} + \tfrac12\chi\,\widetilde X^{-4}\,\widetilde F^i_{(2)}\wedge \widetilde F^i_{(2)}\,,
\end{aligned}\tag{21}$$

$$\begin{aligned}
d(X^4\,{*d\chi}) &= 4g^2\,\chi\,X^2\,\epsilon_{(4)} + \chi\,X^6\,q^{-4}\,{*\widetilde F^i_{(2)}}\wedge \widetilde F^i_{(2)} \\
&\quad -\tfrac12 F^i_{(2)}\wedge F^i_{(2)} + \tfrac12(1-\chi^2\,X^4)\,\widetilde X^{-4}\,\widetilde F^i_{(2)}\wedge \widetilde F^i_{(2)}\,,
\end{aligned}\tag{22}$$

where we are using the functions q and $\widetilde X$ defined in the previous section.

The Yang-Mills equations of motion are

$$\begin{aligned}
D(X^{-2}\,{*F^i_{(2)}}) &= -d\chi\wedge F^i_{(2)}\,, \\
\widetilde D(\widetilde X^{-2}\,{*\widetilde F^i_{(2)}}) &= d(\chi\,X^2\,\widetilde X^{-2})\wedge \widetilde F^i_{(2)}\,.
\end{aligned}\tag{23}$$

where D and $\widetilde D$ are the Yang-Mills-covariant exterior derivatives for the two $SU(2)$ gauge groups:

$$\begin{aligned}
D\,F^i_{(2)} &\equiv dF^i_{(2)} + g\,\epsilon_{ijk}\,A^j_{(1)}\wedge F^k_{(2)} = 0\,, \\
\widetilde D\,\widetilde F^i_{(2)} &\equiv d\widetilde F^i_{(2)} + g\,\epsilon_{ijk}\,\widetilde A^j_{(1)}\wedge \widetilde F^k_{(2)} = 0\,,
\end{aligned}$$

etc. Finally, the Einstein equation is

$$\begin{aligned} R_{\mu\nu} &= \tfrac{1}{2}\partial_\mu\phi\,\partial_\nu\phi + \tfrac{1}{2}e^{2\phi}\,\partial_\mu\chi\,\partial_\nu\chi + \tfrac{1}{2}X^{-2}\,(F^i_{\mu\rho}\,F^{i\,\rho}_\nu - \tfrac{1}{4}(F^i_{(2)})^2\,g_{\mu\nu}) \\ &\quad + \tfrac{1}{2}\widetilde{X}^{-2}\,(\widetilde{F}^i_{\mu\rho}\,\widetilde{F}^{i\,\rho}_\nu - \tfrac{1}{4}(\widetilde{F}^i_{(2)})^2\,g_{\mu\nu})\,. \end{aligned} \tag{24}$$

Note that the Z_2 symmetry (15) can be seen in the Lagrangian (17). It can be made manifest by making use of the $\widetilde{X}$ variable, to rewrite (17) as

$$\begin{aligned} \mathcal{L}_4 &= R\,{*\mathbb{1}} - \tfrac{1}{2}{*d\phi}\wedge d\phi - \tfrac{1}{2}e^{2\phi}\,{*d\chi}\wedge d\chi - V\,{*\mathbb{1}} \\ &\quad -\tfrac{1}{2}X^{-2}\,{*F^i_{(2)}}\wedge F^i_{(2)} - \tfrac{1}{2}\widetilde{X}^{-2}\,{*\widetilde{F}^i_{(2)}}\wedge \widetilde{F}^i_{(2)}\,, \\ &\quad -\tfrac{1}{2}\chi\,F^i_{(2)}\wedge F^i_{(2)} + \tfrac{1}{2}\chi\,X^2\,\widetilde{X}^{-2}\widetilde{F}^i_{(2)}\wedge \widetilde{F}^i_{(2)}\,, \end{aligned} \tag{25}$$

where the potential V can be written as

$$V = -2g^2\,(4 + X^2 + \widetilde{X}^2)\,, \tag{26}$$

4 Reduction from $D = 11$ to $D = 4$

In section 2 we presented our results for the Ansätze for the metric tensor and 4-form field strength of eleven-dimensional supergravity, which, when substituted into the eleven-dimensional equations of motion, give rise to the equations of motion for the four dimensional $N = 4$ $SO(4)$ gauged supergravity. We arrived at the Ansatz for the metric by a combination of generalisation from the previously known abelian case in [12], and the general formula presented in [10], as described in Appendix A. Our procedure for determining the 4-form field strength Ansatz consisted of a combination of generalisation from the abelian case in [12], together with a trial and error process of introducing additional terms as necessary until consistency was achieved. Thus our criterion for determining the Ansatz was to verify explicitly that substituting it into the eleven-dimensional equations of motion gave a consistent reduction to the four-dimensional equations of motion. (In particular, the most non-trivial part is finding an Ansatz that is *consistent*, in the sense that all the dependence on the ξ coordinate of the 7-sphere cancels out in all the equations.)

We shall not present all the details of the substitution of the Ansatz here, because the procedure is an involved one, and in fact parts of it were most conveniently checked by computer. However, it is useful to summarise the structure of the calculation. The $D = 11$ equations of motion, and the Bianchi identity for $\hat{F}_{(4)}$, are given by

$$\hat{R}_{MN} = \tfrac{1}{12}(\hat{F}^2_{MN} - \tfrac{1}{12}\hat{F}^2_{(4)}\,\hat{g}_{MN})\,,$$

$$
\begin{aligned}
d\hat{*}\hat{F}_{(4)} &= -\tfrac{1}{2}\hat{F}_{(4)}\wedge\hat{F}_{(4)}\,, \\
d\hat{F}_{(4)} &= 0\,.
\end{aligned}
\tag{27}
$$

Considering first the Bianchi identity, it is evident from (5), (9) and (10) that since the four-dimensional Hodge duals $*dX$, $*d\chi$, $*F^i_{(2)}$ and $*\widetilde{F}^i_{(2)}$ appear in the Ansatz for $F_{(4)}$, it must be that $d\hat{F}_{(4)}=0$ will not be satisfied as an *identity*, but rather it will imply certain of the four-dimensional equations of motion. Specifically, $d\hat{F}_{(4)}=0$ implies the $D=4$ Yang-Mills equations, and a particular combination of the equations of motion for the dilaton and the axion.

The $D=11$ field equation for $\hat{F}_{(4)}$ gives rise separately to the four-dimensional equations of motion for the dilaton, the axion, and the Yang-Mills fields. Finally, the various components of the eleven-dimensional Einstein equation give rise again to the four-dimensional dilaton, axion and Yang-Mills equations, and also the four-dimensional Einstein equation.

Two comments are in order. Firstly, we remark that, as always in these examples of non-trivial consistent sphere reductions, the consistency is achieved only because of remarkable "conspiracies" between the contributions from the metric and the antisymmetric tensor in the higher-dimensional theory. Thus in this case it is only because of the precise field content, and the structure, of the eleven-dimensional theory that it is possible to obtain a consistent reduction Ansatz in which all the dependence on the coordinates of the internal 7-sphere cancels out when the Ansatz is substituted into the eleven-dimensional equations of motion.

The second comment is that, as in most of the other cases of consistent sphere reductions, we see here also that the Ansatz for the antisymmetric tensor must be made on the *field strength* $\hat{F}_{(4)}$, rather than on the fundamental potential $\hat{A}_{(3)}$. This is evident from the fact that the four-dimensional Hodge duals of dX, $d\chi$, $F^i_{(2)}$ and $\widetilde{F}^i_{(2)}$ appear in the Ansatz (5), (10) (as well as the undualised fields). As we remarked above, this means that the Bianchi identity for $\hat{F}_{(4)}$ is not satisfied *identically*, but rather as a consequence of the four-dimensional equations of motion. Consequently, there is no way to write an explicit Ansatz for the potential $\hat{A}_{(3)}$, since if we could, $d\hat{F}_{(4)}=0$ would be a true identity. The upshot from this is that the Kaluza-Klein sphere reduction must necessarily be discussed at the level of the higher-dimensional equations of motion; it is not possible to describe the reduction at the level of substituting an Ansatz into the higher-dimensional Lagrangian.

5 Gauge-coupling limits

In this section, we address two main topics. Firstly, we show how in the standard $N=4$ $SO(4)$ gauged supergravity, the case with independent $SU(2)$ coupling constants g and $\tilde{g}$ can be derived from the the case where $g=\tilde{g}$, just by field redefinitions. Thus there is really no greater generality when the coupling constants are unequal, in the standard $SO(4)$ gauged theory. It is, however, useful to introduce the artificial extra parameter for the purpose of discussing singular limits. In the second part of this section, we first observe that if one or other of the $SU(2)$ coupling constants is set to zero in the standard $SO(4)$ gauged supergravity, the theory becomes equivalent to the the similar limit of the Freedman-Schwarz $SU(2)\times SU(2)$ gauged theory. Then, we show a more surprising result, which is that the full Freedman-Schwarz theory with g and $\tilde{g}$ both non-zero can in fact be derived as a singular limit of the standard $SO(4)$ gauged theory. This is a limit where the axion χ is shifted by an infinite constant, accompanied by appropriate constant rescalings of certain other fields and coupling constants. We show how in this limit, the previous S^7 reduction Ansatz reduces to an $\mathbb{R}\times S^3\times S^3$ reduction, which can be interpreted as an $S^3\times S^3$ reduction from $D=10$. This makes contact with previous results [25, 26] for obtaining the Freedman-Schwarz model by Kaluza-Klein reduction.

5.1 $N=4$ $SO(4)$ gauged supergravity with $g\neq\tilde{g}$ from $g=\tilde{g}$

To begin, let us show how we can restore the two independent gauge coupling constants g and $\tilde{g}$ in the $N=4$ $SO(4)$ gauged theory, one for each $SU(2)$ factor in the gauge group. To do this, we take the Lagrangian (17), and make the following field and coupling constant redefinitions:

$$\begin{aligned}
&\phi=\phi'+k\,, \qquad \chi=\chi'\,e^{-k}\,, \qquad A^i_{(1)}={A'}^i_{(1)}\,e^{\frac{1}{2}k}\,, \qquad \widetilde{A}^i_{(1)}=\tilde{A}'^i_{(1)}\,e^{-\frac{1}{2}k}\,,\\
&g'=g\,e^{\frac{1}{2}k}\,, \qquad \tilde{g}'=g\,e^{-\frac{1}{2}k}\,,
\end{aligned} \tag{28}$$

where k is a constant. Now dropping the primes, we find that the Lagrangian takes the identical form (17), where now (18) and (19) have become

$$V=-8g\,\tilde{g}\,-2g^2\,e^{\phi}-2\tilde{g}^2\,e^{-\phi}-2\tilde{g}^2\,\chi^2\,e^{\phi}\,, \tag{29}$$

and

$$F^i_{(2)}=dA^i_{(1)}+\tfrac{1}{2}g\,\epsilon_{ijk}\,A^j_{(1)}\wedge A^k_{(1)}\,, \qquad \widetilde{F}^i_{(2)}=d\widetilde{A}^i_{(1)}+\tfrac{1}{2}\tilde{g}\,\epsilon_{ijk}\,\widetilde{A}^j_{(1)}\wedge\widetilde{A}^k_{(1)}\,. \tag{30}$$

The potential (29) can also be rewritten in various equivalent ways:

$$V \;=\; -8g\,\tilde{g}-2g^2\,X^2-2\tilde{g}^2\,\widetilde{X}^2\,,$$

$$
\begin{aligned}
&= -8g\,\tilde{g} - 2(g^2+\tilde{g}^2)\,\cosh\lambda - 2(g^2-\tilde{g}^2)\,\cos\sigma\,\sinh\lambda\,, \qquad (31)\\
&= -\frac{1}{1-|W|^2}\Big(g_+^2\,(3-|W|^2) - g_-^2\,(1-3|W|^2) - 4g_+\,g_-\,A\Big)\,.
\end{aligned}
$$

In the second line, we are using the parametrisation of the scalar fields given by (52) in Appendix A. In the final line, we have written the potential in terms of the complex field $W = -A + \mathrm{i}\,B$ used in [22], which is related to our σ and λ by $W = e^{\mathrm{i}\,\sigma}\,\tanh\frac{1}{2}\lambda$, and coupling constants $g_\pm = g \pm \tilde{g}$.

Thus after the rescaling (28), the standard $N = 4$ $SO(4)$ gauged supergravity with $g = \tilde{g}$ is mapped into the formulation with two independent gauge coupling constants that was presented in [22]. (This was also observed in [23, 24].) As we have seen, it is in fact identical, modulo field redefinitions, to the original theory obtained in [20] where the two gauge coupling constants were equal. (It is easy to see from (31) how this equivalence can pass unnoticed if one uses the (σ, λ) or $W = -A + \mathrm{i}\,B$ parametrisation for the scalar fields.) It is, of course, trivial to substitute the rescalings into the metric and 4-form Ansätze given in section 2, to obtain reduction Ansätze where the two gauge coupling constants are different.

5.2 Freedman-Schwarz as a limit of $N = 4$ $SO(4)$ gauged supergravity

Having obtained the $N = 4$ $SO(4)$ gauged supergravity with independent $SU(2)$ coupling constants g and $\tilde{g}$, the possibility of taking interesting singular limits arises. First, we may consider the situation where we set $\tilde{g} = 0$, whereupon the potential V becomes

$$V = -2g^2\,e^{\phi}\,. \qquad (32)$$

Of course the $SU(2)$ gauge fields $\widetilde{A}^i_{(1)}$ just become abelian $U(1)^3$ in this limit. The theory in this limit is equivalent to the limit of the Freedman-Schwarz model in which one of its $SU(2)$ gauge coupling constants is also set to zero. (This observation was also made in [22].) The equivalence can be made explicit by dualising the fields $\widetilde{A}^i_{(1)}$ (which can now be done because they are abelian). This gives precisely the $\tilde{g} = 0$ limit of the Freedman-Schwarz model (see equation (33) below). Note that instead taking the limit where $g = 0$ rather than $\tilde{g} = 0$ is equivalent, after a field redefinition.

One might now wonder if it could be possible to obtain the complete Freedman-Schwarz model, with both gauge coupling constants non-zero, as a suitable limit of the standard $N = 4$ $SO(4)$ gauged supergravity given by (17). As we shall now show, this is indeed the case. Let us first present the bosonic Lagrangian for the Freedman-Schwarz theory [21]:

$$\mathcal{L}_4^{FS} = R\,{*\mathbb{1}} - \tfrac{1}{2}{*d\phi}\wedge d\phi - \tfrac{1}{2}e^{2\phi}\,{*d\chi}\wedge d\chi - V\,{*\mathbb{1}} \qquad (33)$$

$$-\tfrac{1}{2}e^{-\phi}\,{*F^i_{(2)}}\wedge F^i_{(2)} - \tfrac{1}{2}e^{-\phi}\,{*\widetilde F^i_{(2)}}\wedge \widetilde F^i_{(2)} - \tfrac{1}{2}\chi\, F^i_{(2)}\wedge F^i_{(2)} - \tfrac{1}{2}\chi\,\widetilde F^i_{(2)}\wedge \widetilde F^i_{(2)}\,,$$

with

$$V = -2(g^2+\tilde g^2)\,e^{\phi}\,, \tag{34}$$

and gauge field strengths given by (30).

A natural attempt to obtain this as a limit from (17) is to redefine the fields and coupling constants in (17), (29) and (30) according to

$$\chi = \chi' + b\,,\qquad \widetilde A^i_{(1)} = b\,\tilde A'^i_{(1)}\,,\qquad \tilde g = \tilde g'\,b^{-1}\,, \tag{35}$$

(with all other fields and constants left unscaled), where b is a constant. Indeed, upon sending b to infinity and dropping the primes, we find that (17) becomes precisely (33) with the potential V given by (34), and the field strengths given by (30).[3] (The effect of reversing the sign of the dilaton coupling from e^{ϕ} to $e^{-\phi}$ in the kinetic term for $\widetilde F^i_{(2)}$, which is normally accomplished by dualisation, is instead achieved here by this singular limiting process.)

Normally, one would say that the standard $N=4$ $SO(4)$ gauged supergravity and the Freedman-Schwarz $N=4$ $SU(2)\times SU(2)$ gauged supergravity are intrinsically inequivalent. This can be understood from the fact that they correspond to gauging two different formulations of the same $N=4$ ungauged theory, which are related by a dualisation involving the gauge fields [27]. The processes of gauging and dualisation do not commute (since one cannot dualise non-abelian Yang-Mills fields), and so the gauged theories can no longer be equivalent. Thus normally, one would say that to "relate" the standard $SO(4)$ gauged theory to the Freedman-Schwarz theory, it would be necessary to ungauge one theory, dualise, and then regauge it.

An intriguing outcome of our work is that there is another way of achieving the same effect, by taking a singular limit. Since the limit *is* singular, one should perhaps still view the two gauged theories as being in some sense inequivalent. However, one theory seems to be "more inequivalent" than the other, since we can derive Freedman-Schwarz as a singular limit of the standard $SO(4)$ gauged theory, but the arrow cannot be reversed.[4]

One can also verify that the supersymmetry transformation laws of the standard $SO(4)$ gauged theory do indeed produce those of the Freedman-Schwarz gauged theory when the

[3]Two ostensibly divergent terms of the form $b\,F^i_{(2)}\wedge F^i_{(2)}$ and $b\,\widetilde F^i_{(2)}\wedge \widetilde F^i_{(2)}$ are actually total derivatives, which can be dropped.

[4]Note that one can also apply a similar singular limiting procedure in other examples, including ordinary ungauged supergravities.

$b \longrightarrow \infty$ limit of (35) is taken. We shall not present all the details here, but just the "extra" terms in the spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ transformation laws, which appear only in the gauged theories. From [22], and using the notation of that paper, these extra terms in the $SO(4)$ gauged theory are

$$\begin{aligned} \delta'\,\bar\psi^i_\mu &= \tfrac{\mathrm{i}}{2}\,\bar\epsilon^i\,\gamma_\mu\,\frac{[g_+ + g_-\,(-A + \mathrm{i}\,\gamma_5\,B)]}{(1-|W|^2)^{\frac12}}\,,\\ \delta'\,\bar\chi^i &= \tfrac{1}{\sqrt2}\,\bar\epsilon^i\,\frac{[g_+\,(A - \mathrm{i}\,\gamma_5\,B) - g_-]}{(1-|W|^2)^{\frac12}}\,, \end{aligned} \tag{36}$$

where A, B and $g_\pm$ were defined in section 5.1. Making the replacements (35), and then sending b to infinity, we find that there are exact cancellations of terms linear in b, which would otherwise have been divergent, leaving an overall finite result, namely

$$\begin{aligned} \delta'\,\bar\psi^i_\mu &= \tfrac{\mathrm{i}}{2}\,e^{\frac12\phi}\,\bar\epsilon^i\,(\tilde g - \mathrm{i}\,g\,\gamma_5)\,\gamma_\mu\,,\\ \delta'\,\bar\chi^i &= \tfrac{1}{\sqrt2}\,e^{\frac12\phi}\,\bar\epsilon^i\,(\tilde g - \mathrm{i}\,g\,\gamma_5)\,. \end{aligned} \tag{37}$$

These are precisely the correct forms of the corresponding "extra" terms in the transformation rules of the Freedman-Schwarz model [21]. The rest of the terms in the transformation rules similarly all map over appropriately.

It is of interest to see what happens to our Kaluza-Klein reduction Ansatz if this limit is taken. For the metric Ansatz (1), we find that as b becomes very large, the metric becomes

$$d\hat s_{11}^2 = (\tfrac12 bX^2)^{\frac23}\Big(ds_4^2 + \frac{2}{g\,\tilde g}\,d\tilde\xi^2 + \tfrac12 g^{-2}\,X^{-2}\sum_i (h^i)^2 + \tfrac12\tilde g^{-2}\,X^{-2}\sum_i(\tilde h^i)^2\Big)\,, \tag{38}$$

where we have defined a new coordinate $\tilde\xi$ by $\xi = b^{-\frac12}\,\tilde\xi + \frac14\pi$. Similarly, we find that the Ansatz for the 4-form field strength, given in (5), (9) and (10), reduces to

$$\begin{aligned} \hat F_{(4)} &= \frac{b}{\sqrt{2g\,\tilde g}}\Big(X^4 {*d\chi}\wedge d\tilde\xi - \tfrac12 g^{-2}\,d\tilde\xi\wedge\epsilon_{(3)} - \tfrac12\tilde g^{-2}\,d\tilde\xi\wedge\tilde\epsilon_{(3)}\\ &\qquad + \tfrac12 g^{-1}\,d\tilde\xi\wedge F^i_{(2)}\wedge h^i + \tfrac12\tilde g^{-1}\,d\tilde\xi\wedge\widetilde F^i_{(2)}\wedge\tilde h^i\Big)\,. \end{aligned} \tag{39}$$

We see that the metric Ansatz has an overall $b^{2/3}$ constant factor, while the 4-form Ansatz has an overall b factor. These in fact precisely cancel out when the Ansätze are substituted into the eleven-dimensional equations of motion (27), since there is a scaling symmetry in $D = 11$ under

$$\hat g_{MN} \longrightarrow e^{2k}\,\hat g_{MN}\,,\qquad \hat A_{MNP} \longrightarrow e^{3k}\,\hat A_{MNP}\,. \tag{40}$$

Thus even though b is being sent to infinity, the Ansatz still gives a sensible limit. In fact using (40), we can effectively set b to any desired value in (38) and (39). It is convenient to take $b = 2$.

The Ansätze (38) and (39) can be reinterpreted as a first reduction step from $D=11$ to $D=10$ on the $\tilde{\xi}$ Killing direction, followed by a reduction on $S^3\times S^3$. To go from $D=11$ to $D=10$ we follow the standard Kaluza-Klein prescription, with

$$\begin{aligned} d\hat{s}_{11}^2 &= e^{-\frac{1}{6}\varphi}\, ds_{10}^2 + e^{\frac{4}{3}\varphi}\,(d\tilde{\xi}+\mathcal{A}_{(1)})^2\,,\\ \hat{F}_{(4)} &= F_{(4)} + F_{(3)}\wedge(d\tilde{\xi}+\mathcal{A}_{(1)})\,, \end{aligned} \tag{41}$$

where $F_{(4)} = dA_{(3)} - dA_{(2)}\wedge\mathcal{A}_{(1)}$ and $F_{(3)} = dA_{(2)}$ (The field-strength reduction follows from $\hat{A}_{(3)} = A_{(3)} + A_{(2)}\wedge d\tilde{\xi}$.) Comparing with (38) and (39), we see that in the $D=10$ type IIA theory we shall have

$$\begin{aligned} ds_{10}^2 &= (\tfrac{2}{g\,\tilde{g}})^{\frac{1}{8}}\Big[e^{\frac{3}{4}\phi}\,ds_4^2 + e^{-\frac{1}{4}\phi}\Big(g^{-2}\sum_i (h^i)^2 + \tilde{g}^{-2}\sum_i (\tilde{h}^i)^2\Big)\Big]\,,\\ F_3 &= \frac{1}{\sqrt{2g\,\tilde{g}}}\Big[2e^{2\phi}\,{*d\chi} + g^{-2}\,\epsilon_{(3)} + \tilde{g}^{-2}\,\tilde{\epsilon}_{(3)} - g^{-1}\,F^i_{(2)}\wedge h^i - \tilde{g}^{-1}\,\widetilde{F}^i_{(2)}\wedge\tilde{h}^i\Big]\,,\\ \varphi &= \tfrac{1}{2}\phi - \tfrac{3}{4}\log(\tfrac{1}{2}g\,\tilde{g})\,,\\ F_{(4)} &= 0\,,\qquad \mathcal{A}_{(1)} = 0\,. \end{aligned} \tag{42}$$

Thus only the NS-NS fields of the type IIA theory are active (the metric, the dilaton φ, and the 3-form field strength $F_{(3)}$), while the R-R fields $F_{(4)}$ and $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$ are zero. The reduction Ansatz (42) can therefore also be interpreted as a reduction in the type I or heterotic string. It is easy to check that it agrees precisely with the reduction given in [25, 26], for obtaining the Freedman-Schwarz model as an $S^3\times S^3$ reduction of the heterotic theory. This singular limit of the S^7 reduction is reminiscent of examples discussed previously in [28].

Once again, the "one-way" nature of the limiting procedure can be seen in the Kaluza-Klein reduction. One can take a singular limit in which S^7 becomes $\mathbb{R}\times S^3\times S^3$, but one cannot reverse the process and obtain S^7 as a limit of $\mathbb{R}\times S^3\times S^3$.

It is interesting to note that no analogue of the scaling (28) arises in the Freedman-Schwarz model. As we showed, in the standard $N=4$ $SO(4)$ gauged theory this scaling means that there is really no distinction between the situation where the gauge coupling constants g and $\tilde{g}$ of the two $SU(2)$ factors in the gauge group are equal or unequal. On the other hand, the absence of such a scaling transformation in the Freedman-Schwarz case means that the ratio between its $SU(2)$ coupling constants g and $\tilde{g}$ is a genuine parameter of the theory, with no redefinition that maps one value into another.

6 Conclusion

In this paper, we have constructed the complete, non-linear, explicit Kaluza-Klein Ansatz for obtaining the bosonic sector of four-dimensional $N = 4$ $SO(4)$ gauged supergravity by dimensional reduction of eleven-dimensional supergravity on S^7. Although in principle subsumed by the $N = 8$ reduction constructed in [10], the advantage of our $N = 4$ truncation is that the resulting four-dimensional theory is much simpler than the maximal $N = 8$ theory, and consequently the reduction Ansatz is much more manageable. In fact it is because of this simplification that we have been able to construct a fully explicit Kaluza-Klein reduction.

The key point, in fact, is that the surviving gauge group in the truncation, namely $SO(4)$, is the product of two $SU(2)$ factors, and the gauge bosons for these factors arise from two separate 3-spheres in the parameterisation of the internal 7-sphere as a foliation of $S^3 \times S^3$ hypersurfaces. Thus we are able to benefit from the fact that the 3-spheres are themselves $SU(2)$ group manifolds. A group manifold G has a $G \times G$ isometry group, comprising independent left-translations and right-translations under G. Since we need only include the gauge bosons associated with the left-translations on each 3-sphere, the corresponding deformations of the 3-spheres preserve their homogeneity. Thus although the 7-sphere itself is distorted inhomogeneously when the lower-dimensional fields are excited, these inhomogeneities are limited to co-dimension 1, corresponding to a distortion of the foliation whilst keeping the $S^3 \times S^3$ surfaces themselves homogeneous. For this reason, the dependence of the Ansatz on the coordinates of the internal 7-sphere is restricted to the "latitude" coordinate ξ that parameterises the foliating surfaces. It is still, of course, highly non-trivial that the overall ξ coordinate dependence cancels out in the eleven-dimensional equations of motion under the Kaluza-Klein reduction.

Using the results obtained in this paper, any bosonic solution of four-dimensional $N = 4$ $SO(4)$ gauged supergravity can be oxidised back to an exact solution of eleven-dimensional supergravity.

We have shown that the $N = 4$ $SO(4)$ gauged supergravity really has only one genuine gauge-coupling parameter, and that although one can introduce "independent" parameters g and $\tilde{g}$ for the two $SU(2)$ gauge groups, this is nothing but a redefinition of fields in the theory with $g = \tilde{g}$. This is a different situation from the Freedman-Schwarz model, where the two gauge couplings are genuinely independent parameters, which cannot be set equal by field redefinitions.

We have also shown that the Freedman-Schwarz model arises as a singular limit of the standard $N = 4$ $SO(4)$ gauged supergravity, in which the axion is shifted by an infinite constant, together with appropriate rescalings of other fields. We have shown how this translates, in our Kaluza-Klein reduction, to a limiting case where the 7-sphere degenerates into the product $\mathbb{R} \times S^3 \times S^3$.

Appendices

A Derivation of the Reduction Ansatz

In this Appendix, we present some of the details of how we arrived at the Kaluza-Klein metric reduction Ansatz that we present in section 2. First, we show how the Ansatz can be deduced, in the absence of the axion, from previous results [12] for the $U(1)^4$ truncation of four-dimensional maximal gauged supergravity. Then, we show how the general results in [10] allow us to obtain the metric Ansatz after the inclusion of the axionic scalar field. The final part of Appendix A comprises a collection of useful *lemmata* for the $SU(2)$-valued forms that are used in the construction of the Ansatz.

A.1 Ansatz with axion set to zero

The structure of the embedding of four-dimensional $N = 4$ gauged $SO(4)$ supergravity in $D = 11$ can be seen by first considering the maximal abelian $U(1)^4$ embedding obtained in [12]. In that case there are three dilatons and three axions in the full $U(1)^4$ theory, although in the reduction Ansatz derived in [12], the axions were set to zero. We can make a further truncation to $U(1)^2$, by setting pairs of the original four $U(1)$ gauge fields equal. At the same time, for consistency, two dilatons and two axions are set to zero. In the axion-free situation described in [12], the metric reduction Ansatz is

$$ds_{11}^2 = \widetilde{\Delta}^{2/3}\, ds_4^2 + g^{-2}\, \widetilde{\Delta}^{-1/3} \sum_i X_i^{-1} \Big(d\mu_i^2 + \mu_i^2\, (d\phi_i + g\, A_{(1)}^i)^2\Big)\,. \tag{43}$$

where $\widetilde{\Delta} = \sum_{i=1}^4 X_i\, \mu_i^2$. The four quantities μ_i satisfy $\sum_i \mu_i^2 = 1$. The four scalars X_i, which satisfy $X_1 X_2 X_3 X_4 = 1$, are parameterised by the three dilatons $\vec{\phi}$, as $X_i = \exp(-\frac{1}{2}\vec{a}_i \cdot \vec{\phi})$, for certain constant 3-vectors $\vec{a}_i$. Setting two of the dilatons to zero leads to $X_1 = X_2 \equiv X$, $X_3 = X_4 = 1/X$. At the same time, the $U(1)$ gauge fields are set pairwise equal, with $A_{(1)}^1 = A_{(1)}^2 \equiv A_{(1)}$, $A_{(1)}^3 = A_{(1)}^4 \equiv \widetilde{A}_{(1)}$. Thus the metric Ansatz (43) reduces to

$$\begin{aligned} d\hat{s}_{11}^2 &= \Delta^{\frac{2}{3}}\, ds_4^2 + 4g^{-2}\, \Delta^{\frac{2}{3}}\, d\xi^2 \\ &\quad + g^{-2}\, \Delta^{-\frac{1}{3}}\, X^{-1}\, c^2 \Big(d\theta^2 + \sin^2\theta\, d\varphi^2 + (d\psi + \cos\theta\, d\varphi - g\, A_{(1)})^2\Big) \end{aligned}$$

$$+g^{-2}\,\Delta^{-\frac{1}{3}}\,X\,s^2\,\Big(d\tilde\theta^2+\sin^2\tilde\theta\,d\tilde\varphi^2+(d\widetilde\psi+\cos\tilde\theta\,d\tilde\varphi-g\,\widetilde A_{(1)})^2\Big)\,, \tag{44}$$

where we have found it convenient to parameterise the four quantities μ_i as

$$\mu_1=c\,\cos\tfrac12\theta\,,\quad \mu_2=c\,\sin\tfrac12\theta\,,\quad \mu_3=s\,\cos\tfrac12\tilde\theta\,,\quad \mu_4=s\,\sin\tfrac12\tilde\theta\,, \tag{45}$$

where $c\equiv\cos\xi$ and $s\equiv\sin\xi$, and the four azimuthal angles ϕ_i as

$$\phi_1=\tfrac12(\psi+\varphi)\,,\quad \phi_2=\tfrac12(\psi-\varphi)\,,\quad \phi_3=\tfrac12(\widetilde\psi+\tilde\varphi)\,,\quad \phi_4=\tfrac12(\widetilde\psi-\tilde\varphi)\,. \tag{46}$$

If we temporarily set $X=1$ and $A_{(1)}=\widetilde A_{(1)}=0$ (*i.e.* turning off the four-dimensional field excitations) , we see that the internal seven-dimensional metric in (44) becomes the round 7-sphere, written as

$$d\Omega_7^2=d\xi^2+\cos^2\xi\,d\Omega_3^2+\sin^2\xi\,d\widetilde\Omega_3^2\,, \tag{47}$$

where $d\Omega_3^2$ and $d\widetilde\Omega_3^2$ are two separate unit 3-sphere metrics, written in terms of the Euler angles (θ,φ,ψ) and $(\tilde\theta,\tilde\varphi,\widetilde\psi)$ respectively.

A natural generalisation of the reduction Ansatz (44) now suggests itself, in which we enlarge the $U(1)$ gauge field in each 3-sphere to $SU(2)$:

$$d\hat s_{11}^2=\Delta^{\frac23}\,ds_4^2+4g^{-2}\,\Delta^{\frac23}\,d\xi^2+g^{-2}\,\Delta^{-\frac13}\,\Big(X^{-1}\,c^2\,\sum_{i=1}^3(h^i)^2+X\,s^2\,\sum_{i=1}^3(\tilde h^i)^2\Big)\,, \tag{48}$$

where

$$h^i\equiv\sigma_i-gA^i_{(1)}\,,\qquad \tilde h^i\equiv\tilde\sigma_i-g\widetilde A^i_{(1)}\,, \tag{49}$$

where σ_i are left-invariant 1-forms on the first S^3, and $\widetilde\sigma_i$ are left-invariant 1-forms on the second S^3.

A.2 Ansatz with non-vanishing axion

In the above, we considered the situation when the axion of the $N=4$ theory is set to zero. When the axion is non-zero, we cannot deduce the form of the metric Ansatz from the previous results in [12]. Now, we can make use of the general formalism in [10], where the reduction Ansatz for the $N=8$ theory was obtained. In particular, the full metric Ansatz in [10] is relatively simple, and by truncating it appropriately we are able to construct the Ansatz for the $N=4$ theory.

To begin, we need to determine the tensors $u_{ij}{}^{IJ}(x)$ and $v_{ijIJ}(x)$ that appear in the definition of the scalar 56-bein $\mathcal V$ and its inverse,

$$\mathcal V=\begin{pmatrix} u_{ij}{}^{IJ} & v_{ijKL}\\ v^{k\ell IJ} & u^{k\ell}{}_{KL}\end{pmatrix}\,,\qquad \mathcal V^{-1}=\begin{pmatrix} u^{ij}{}_{IJ} & -v_{k\ell IJ}\\ -v^{ijKL} & u_{k\ell}{}^{KL}\end{pmatrix}\,. \tag{50}$$

In the $N = 4$ gauged $SU(2) \times SU(2)$ truncation of the full $N = 8$ gauged $SO(8)$ theory, we find that these are given by

$$
\begin{aligned}
&u_{ab}{}^{cd} = 2\cosh \tfrac{1}{2}\lambda\, \delta^{cd}_{ab}\,, \qquad u_{\bar a\bar b}{}^{\bar c\bar d} = 2\cosh \tfrac{1}{2}\lambda\, \delta^{\bar c\bar d}_{\bar a\bar b}\,, \qquad u_{a\bar b}{}^{c\bar d} = 2\delta^c_a\, \delta^{\bar d}_{\bar b}\,,\\
&v_{abcd} = \sinh \tfrac{1}{2}\lambda\, e^{\mathrm{i}\sigma}\, \epsilon_{abcd}\,, \qquad v_{\bar a\bar b\bar c\bar d} = \sinh \tfrac{1}{2}\lambda\, e^{-\mathrm{i}\sigma}\, \epsilon_{\bar a\bar b\bar c\bar d}\,,
\end{aligned}
\tag{51}
$$

where we have split the indices $i = (1, 8)$ into $i = (a, \bar a)$, where $a = (1, 4)$ and $\bar a = (5, 8)$, and similarly for I. The fields λ and σ are related to the usual dilaton ϕ and axion χ by

$$
\begin{aligned}
\cosh\lambda &= \cosh\phi + \tfrac{1}{2}\chi^2\, e^{\phi}\,,\\
\cos\sigma\, \sinh\lambda &= \sinh\phi - \tfrac{1}{2}\chi^2\, e^{\phi}\,,\\
\sin\sigma\, \sinh\lambda &= \chi\, e^{\phi}\,.
\end{aligned}
\tag{52}
$$

(This is the mapping from the metric $ds_2^2 = d\lambda^2 + \sinh^2\lambda\, d\sigma^2$ to $ds_2^2 = d\phi^2 + e^{2\phi}\, d\chi^2$. Note that in terms of σ and λ, the scalar potential (26) is simply given by $V = -4g^2\,(\cosh\lambda + 2)$.)

It is shown in [9, 10] that the Ansatz for the inverse metric in the internal space (the 7-sphere) is

$$
\hat\Delta(x,y)\, g^{mn}(x,y) = \tfrac{1}{2}(K^{mIJ}\, K^{nKL} + K^{nIJ}\, K^{mKL})\, (u_{ij}{}^{IJ} + v_{ijIJ})\, (u^{ij}{}_{KL} + v^{ijKL})\,, \tag{53}
$$

where

$$
\hat\Delta^2 = \frac{\det(g_{mn}(x,y))}{\det(\bar g_{mn}(y))}\,. \tag{54}
$$

Here $\bar g_{mn}(y)$ denotes the metric of the undistorted round 7-sphere, and K^{mIJ} are the 28 Killing vectors in this metric. Substituting our expressions (51) into (53), we find

$$
\begin{aligned}
\hat\Delta\, \bar g^{mn}(x,y) \;=\;& \sum_{i,j} K^{m\,ij}\, K^{n\,ij} + \tfrac{1}{4}(X^2 - 1) \sum_{\alpha=1}^{3} \Big((J^\alpha_{ab}\, K^{m\,ab})^2 + (J^\alpha_{\bar a\bar b}\, K^{m\,\bar a\bar b})^2 \Big)\\
&+ \tfrac{1}{4}(\widetilde X^2 - 1) \sum_{\alpha=1}^{3} \Big((\widetilde J^\alpha_{ab}\, K^{m\,ab})^2 + (\widetilde J^\alpha_{\bar a\bar b}\, K^{m\,\bar a\bar b})^2 \Big)\,,
\end{aligned}
\tag{55}
$$

where

$$
\begin{aligned}
&J^1_{12} = J^1_{34} = J^2_{13} = -J^2_{24} = J^3_{14} = J^3_{23} = 1\,,\\
&J^1_{56} = J^1_{78} = J^2_{57} = -J^2_{68} = J^3_{58} = J^3_{67} = 1\,,\\
&\widetilde J^1_{12} = -\widetilde J^1_{34} = \widetilde J^2_{13} = \widetilde J^2_{24} = \widetilde J^3_{14} = -\widetilde J^3_{23} = 1\,,\\
&\widetilde J^1_{56} = -\widetilde J^1_{78} = \widetilde J^2_{57} = \widetilde J^2_{68} = \widetilde J^3_{58} = -\widetilde J^3_{67} = 1\,.
\end{aligned}
\tag{56}
$$

Thus J^α_{ab} and $J^\alpha_{\bar a\bar b}$ are self-dual in the $4 + 4$ dimensional subspaces spanned by $i = (a, \bar a)$, while $\widetilde J^\alpha_{ab}$ and $\widetilde J^\alpha_{\bar a\bar b}$ are anti-self-dual.

It is easy to see that the 3 Killing vector combinations $K^{m\,\alpha} \equiv J^\alpha_{ab}\, K^{m\,ab}$ and the 3 combinations $\bar K^{m\,\alpha} \equiv J^\alpha_{\bar a\bar b}\, K^{m\,\bar a\bar b}$ each close on $SU(2)$, and that the two sets mutually commute. Likewise, the 3 combinations $\widetilde K^{m\,\alpha} \equiv \widetilde J^\alpha_{ab}\, K^{m\,ab}$ and the 3 combinations $\widetilde{\bar K}^{m\,\alpha} \equiv \widetilde J^\alpha_{\bar a\bar b}\, K^{m\,\bar a\bar b}$ each close on $SU(2)$, and these commute with each other and the other two $SU(2)$'s. In fact what we are seeing here are the sets of $3+3$ Killing vectors on each of two 3-spheres: $K^{m\,\alpha}$ and $\bar K^{m\,\alpha}$ are the left-translation and right-translation Killing vectors of one 3-sphere, while $\widetilde K^{m\,\alpha}$ and $\widetilde{\bar K}^{m\,\alpha}$ are the left-translation and right-translation Killing vectors of the other 3-sphere.

Now, considering the first S^3, the sum of the squares of the left-translation Killing vectors, $K^{m\,\alpha}\, K^{n\,\alpha}$ is equal to the sum of the squares of the right-translation Killing vectors, $\bar K^{m\,\alpha}\, \bar K^{n\,\alpha}$, each sum giving the bi-invariant inverse metric g_3^{mn} on the S^3. A similar remark applies to the second S^3. Also, the sum of the squares of all 28 Killing vectors gives the inverse metric on the round 7-sphere, so (55) becomes

$$\hat\Delta\, \bar g^{mn}(x,y) = \bar g^{mn}(y) + (X^2-1)\, g_3^{mn}(y) + (\widetilde X^2 - 1)\, \tilde g_3^{mn}(y)\,. \tag{57}$$

This metric is easily inverted, and in terms of the representation $d\Omega_7^2$ in (47) for the round 7-sphere metric, and $d\Omega_3^2 = \frac14 \sum_{i=1}^3 \sigma_i^2$ and $d\widetilde\Omega_3^2 = \frac14 \sum_{i=1}^3 \tilde\sigma_i^2$ for the two round 3-sphere metrics, we obtain

$$ds_7^2 = \hat\Delta^{-1}\Big(d\xi^2 + \tfrac14 \frac{c^2}{c^2\, X^2 + s^2}\, \sum_{i=1}^3 \sigma_i^2 + \tfrac14 \frac{s^2}{s^2\, \widetilde X^2 + c^2}\, \sum_{i=1}^3 \tilde\sigma_i^2\Big)\,, \tag{58}$$

where, as usual, $c=\cos\xi$ and $s=\sin\xi$. From the expression in [9, 10] for the eleven-dimensional metric in terms of the seven-dimensional one, $d\hat s_{11}^2 = \widetilde\Delta^{-1}\, ds_4^2 + ds_7^2$, and noting that our factor Δ is related to the corresponding factor $\hat\Delta$ of [9, 10] by $\Delta = \hat\Delta^{-3/2}$, we eventually arrive at our metric Ansatz (1), after introducing the $SO(4) = SU(2)\times SU(2)$ gauge fields as described earlier. Note that it reduces to (48) if we set the axion to zero. (We have introduced the gauge coupling constant by means of appropriate rescalings.)

In order to check the consistency of the reduction Ansätze presented in section 2, a necessary ingredient is the calculation of the Ricci tensor for the metric Ansatz (1). If we define

$$e^\beta \equiv \Delta^{\frac13}\,,\qquad e^\gamma \equiv (\sqrt2\, g)^{-1}\, c\, \Delta^{\frac13}\, \Omega^{-\frac12}\,,\qquad e^{\tilde\gamma} \equiv (\sqrt2\, g)^{-1}\, s\, \Delta^{\frac13}\, \widetilde\Omega^{-\frac12}\,, \tag{59}$$

then a natural orthonormal basis is

$$\hat e^a = e^\beta\, e^a\,,\qquad \hat e^0 = \sqrt2\, g^{-1}\, e^\beta\, d\xi\,,\quad \hat e^i = e^\gamma\, h^i\,,\quad \hat e^{\tilde i} = e^{\tilde\gamma}\, \tilde h^i\,. \tag{60}$$

In terms of this basis, we find that the vielbein components of the Ricci tensor are given by

$$
\begin{aligned}
\hat R_{00} &= \Delta^{-\frac{2}{3}}\Big[-\Box\beta+\beta_a\,\beta_a+\tfrac12 g^2(-4\beta''-3\gamma''-3\tilde\gamma''+3\beta'\,\gamma'+3\beta'\,\tilde\gamma'-3\gamma'^2-3(\tilde\gamma')^2)\Big]\,,\\
\hat R_{0a} &= \tfrac{3}{\sqrt2}g\,\Delta^{-\frac{2}{3}}\Big[(\beta_a-\gamma_a)\,\gamma'+(\beta_a-\tilde\gamma_a)\,\tilde\gamma'\Big]\,,\\
\hat R_{0i} &= 0\,,\qquad \hat R_{0\tilde i}=0\,,\\
\hat R_{ab} &= \Delta^{-\frac{2}{3}}\Big[R_{ab}-3(\beta_a\,\beta_b+\gamma_a\,\gamma_b+\tilde\gamma_a\,\tilde\gamma_b)+(-\Box\beta+2\beta_c\,\beta_c)\,\eta_{ab}\\
&\quad -\tfrac14 c^2\,\Omega^{-1}\,F^i_{ac}\,F^i_{bc}-\tfrac14 s^2\,\widetilde\Omega^{-1}\,\widetilde F^i_{ac}\,\widetilde F^i_{bc}-\tfrac12 g^2(\beta''+6\beta'\,\cot 2\xi)\,\eta_{ab}\Big]\,,\\
\hat R_{ai} &= -\tfrac{1}{2\sqrt2}\,c\,\Delta^{-\frac23}\,\Omega^{-\frac12}\,[D_b\,F^i_{ab}-2(\beta_b-\gamma_b)\,F^i_{ab}]\,,\\
\hat R_{a\tilde i} &= -\tfrac{1}{2\sqrt2}\,s\,\Delta^{-\frac23}\,\widetilde\Omega^{-\frac12}\,[\widetilde D_b\,\widetilde F^i_{ab}-2(\beta_b-\tilde\gamma_b)\,\widetilde F^i_{ab}]\,,\\
\hat R_{ij} &= \Delta^{-\frac23}\Big[\tfrac12 g^2\,(-\gamma''-6\gamma'\,\cot 2\xi+2\Omega\,c^{-2})\,\delta_{ij}-\Box\gamma\,\delta_{ij}+\tfrac18 c^2\,\Omega^{-1}\,F^i_{ab}\,F^j_{ab}\Big]\,,\\
\hat R_{\tilde i\tilde j} &= \Delta^{-\frac23}\Big[\tfrac12 g^2\,(-\tilde\gamma''-6\tilde\gamma'\,\cot 2\xi+2\widetilde\Omega\,s^{-2})\,\delta_{ij}-\Box\tilde\gamma\,\delta_{ij}+\tfrac18 s^2\,\widetilde\Omega^{-1}\,\widetilde F^i_{ab}\,\widetilde F^j_{ab}\Big]\,,\\
\hat R_{i\tilde j} &= \tfrac18 s\,c\,\Delta^{-\frac53}\,F^i_{ab}\,\widetilde F^j_{ab}\,.
\end{aligned}
\qquad (61)
$$

Note that here β_a, γ_a and $\tilde\gamma_a$ denote the vielbein components of the four-dimensional spacetime derivatives of these functions, so $\beta_a=\partial_a\,\beta$, *etc.* Similarly, β', γ' and $\tilde\gamma'$ denote their derivatives with respect to ξ. Useful identities are $\beta_a+\gamma_a+\tilde\gamma_a=0$, and $\beta'+\gamma'+\tilde\gamma'=2\cot 2\xi$.

Some partial formulae for the 4-form Ansatz are presented in [10], but it is difficult to turn them into explicit expressions, and in any case not all components are presented. We therefore determined the 4-form Ansatz in section 2 by brute-force methods.

A.3 Some $SU(2)$ Lemmata

Here, we collect together some useful properties of the $SU(2)$-valued forms that arise in the reduction Ansatz. We define $h^i\equiv\sigma_i-g\,A^i_{(1)}$, and $\tilde h^i\equiv\tilde\sigma_i-g\,\widetilde A^i_{(1)}$, where σ_i and $\tilde\sigma_i$ are sets of left-invariant 1-forms on the two 3-spheres, satisfying (3). The Yang-Mills field strengths are defined by (19). From their Bianchi identities, we see that we should define gauge-covariant exterior derivatives

$$
D\omega^i\equiv d\omega^i+g\,\epsilon_{ijk}\,A^j_{(1)}\wedge\omega^k\,,\qquad \widetilde D\tilde\omega^i\equiv d\tilde\omega^i+g\,\epsilon_{ijk}\,\widetilde A^j_{(1)}\wedge\tilde\omega^k\,, \qquad (62)
$$

for any forms with an adjoint index of the untilded or tilded $SU(2)$. The Bianchi identities themselves are then $D\,F^i_{(2)}=0$, $\widetilde D\,\widetilde F^i_{(2)}=0$.

We can now derive a number of lemmata. Since every formula for the untilded $SU(2)$ has an identical companion formula for the tilded $SU(2)$, we shall just present the untilded ones.

$$
dh^i=-\tfrac12\epsilon_{ijk}\,h^j\wedge k^k-g\,F^i_{(2)}-g\,\epsilon_{ijk}\,A^j_{(1)}\wedge h^k\,. \qquad (63)
$$

This can be written more elegantly using the gauge-covariant exterior derivative defined in (62):

$$Dh^i = -\tfrac{1}{2}\epsilon_{ijk}\, h^j \wedge h^k - g\, F^i_{(2)}\,. \tag{64}$$

This is a convenient way to express the result, because the gauge-covariant exterior derivative respects the Leibniz rule, just as the ordinary exterior derivative does. Thus, for example,

$$\begin{aligned} d(h^i \wedge F^i_{(2)}) &= Dh^i \wedge F^i_{(2)} - h^i \wedge DF^i_{(2)} = Dh^i \wedge F^i_{(2)} \\ &= -\tfrac{1}{2}\epsilon_{ijk}\, h^j \wedge h^k \wedge F^i_{(2)} - g\, F^i_{(2)} \wedge F^i_{(2)}\,. \end{aligned} \tag{65}$$

Another result is

$$d(h^i \wedge {*F^i_{(2)}}) = -\tfrac{1}{2}\epsilon_{ijk}\, h^j \wedge h^k \wedge {*F^i_{(2)}} - g\, {*F^i_{(2)}} \wedge F^i_{(2)} - h^i \wedge D{*F^i_{(2)}}\,. \tag{66}$$

It is also useful to derive that

$$D(\epsilon_{ijk}\, h_j \wedge h_k) = -2\epsilon_{ijk}\, h^j \wedge Dh^k = 2g\, \epsilon_{ijk}\, h^j \wedge F^k_{(2)}\,. \tag{67}$$

From this, we see, for example, that

$$d(\epsilon_{ijk}\, h_i \wedge h_j \wedge F^k_{(2)}) = 0\,, \qquad d(\epsilon_{ijk}\, h_i \wedge h_j \wedge {*F^k_{(2)}}) = \epsilon_{ijk}\, h_i \wedge h_j \wedge D{*F^k_{(2)}}\,. \tag{68}$$

We also have the result that with $\epsilon_{(3)} \equiv h^1 \wedge h^2 \wedge h^3$,

$$d\epsilon_{(3)} = -\tfrac{1}{2} g\, \epsilon_{ijk}\, h^i \wedge h^j \wedge F^k_{(2)}\,. \tag{69}$$

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